Solutions

Exam 2
Chapters 14 and 15

Name:________________________

Do not write your name on any other page. Answer the following questions. *Answers without proper evidence of knowledge will not be given credit.* Make sure to make reasonable simplifications. Do not approximate answers. Give exact answers. **No calculators are allowed on this exam.**

Show your work!

1. (5 points each)
Find the domain of the following multi-variable functions.

(a) $f(x, y) = \frac{\sqrt{1-x^2-y^2}}{3}$.

$\text{Dom} f = \{(x,y) : x^2+y^2 \le 1\}$

(b) $g(x, y, z) = \frac{\ln z}{x^2+y^2}$.

$\text{Dom} g = \{(x,y,z) : x^2+y^2 \ne 0,\ z > 0\}$

2. Let $z = x^2 \sin xy$ and find the differential dz. (If you don't remember what dz is you can find $\partial z/\partial x$ and $\partial z/\partial y$ for partial credit.) (10 points)

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$= (2x \sin xy + x^2 y \cos xy)dx + x^3 \cos xy \, dy$$

3. Let $z = f(x, y)$ be differentiable and let

$$x = g(t) \quad y = h(t) \quad g(3) = 2 \quad h(3) = 7$$

$$g'(3) = 5 \quad h'(3) = -4 \quad f_x(2, 7) = 6 \quad f_y(2, 7) = -8.$$

Find dz/dt when $t = 3$. (10 points)

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= f_x(g(3), h(3))\, g'(3) + f_y(g(3), h(3))\, h'(3)$$

$$= 6 \cdot 5 + (-8)(-4) = 62$$

4. Let $z = e^{x+2y}$, $x = s/t$, and $y = t/s$. Find $\partial z/\partial s$ and $\partial z/\partial t$. (10 points)

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= e^{x+2y} \cdot \frac{1}{t} + 2e^{x+2y} \cdot \frac{-t}{s^2} = e^{x+2y}\left(\frac{1}{t} - \frac{2t}{s^2}\right)$$

$$= e^{s/t + 2t/s}\left(\frac{1}{t} - \frac{2t}{s^2}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} :$$

$$= e^{x+2y} \cdot \frac{-s}{t^2} + 2e^{x+2y} \cdot \frac{1}{s} = e^{x+2y}\left(\frac{-s}{t^2} + \frac{2}{s}\right)$$

$$= e^{s/t + 2t/s}\left(-\frac{s}{t^2} + \frac{2}{s}\right)$$

5. The temperature at a point (x, y, z) is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$$

where T is measured in $°C$ and x, y, z in meters.

(a) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction toward the point $Q(3, -3, 3)$. (10 points)

(b) In which direction does the temperature increase the fastest at P? (5 points)

(c) Find the maximum rate of change at P. (5 points)

(a) $\vec{PQ} = \langle 1, -2, 1 \rangle \qquad u = \dfrac{\vec{PQ}}{\|\vec{PQ}\|} = \dfrac{1}{\sqrt{6}} \langle 1, -2, 1 \rangle$

$$\nabla T = 200 \langle -2x e^{-x^2-3y^2-9z^2}, -6y e^{-x^2-3y^2-9z^2}, -18z e^{-x^2-3y^2-9z^2} \rangle$$

$$\nabla T(2,-1,2) = 200 \langle -4e^{-43}, 6e^{-43}, -36e^{-43} \rangle$$

$$D_u T(2,-1,2) = \frac{200e^{-43}}{\sqrt{6}} \langle -4, 6, -36 \rangle \cdot \langle 1, -2, 1 \rangle$$

$$= \frac{200e^{-43}}{\sqrt{6}} (-4-12-36) = \frac{-10{,}400e^{-43}}{\sqrt{6}}$$

(b) Same direction as $\nabla T(2,-1,2)$.

(c) $D_u T(2,-1,2)$ is maximized when u & ∇T face same way.

$$D_u T(2,-1,2) = |\nabla T(2,-1,2)| \cdot |u| \cdot \cos\theta$$

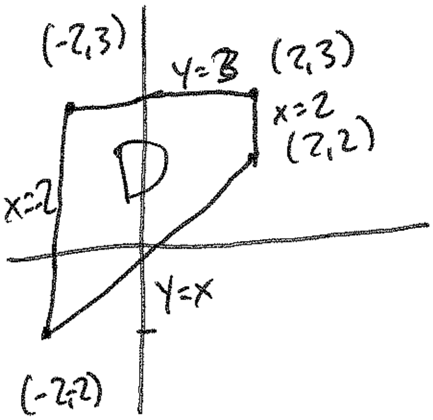
$$= |\nabla T(2,-1,2)|$$

$$= 400e^{-43} |\langle -2, 3, -18 \rangle|$$

$$= 400e^{-43} \sqrt{4+9+324}$$

$$= 400e^{-43} \sqrt{337}$$

6. Find the absolute maximum and minimum of $f(x, y) = x^3 - 3x - y^3 + 12y$ on the quadrilateral whose vertices are $(-2, 3), (2, 3), (2, 2)$, and $(-2, -2)$. (15 points)



(1) Find max & min on interior

$f_x = 3x^2 - 3$, $f_y = 12 - 3y^2$

$f_x = 0 \Rightarrow x^2 = 1$ $\quad f_y = 0 \Rightarrow y^2 = 4$

Crit. pts inside D are $(-1,2), (1,2)$

$f(-1,2) = 18 \quad f(1,2) = 14$

(2) Find max & min on boundaries.

(a) Test all corners: $f(-2,-2) = -18 \quad f(-2,3) = 7$

$f(2,2) = 18 \quad f(2,3) = 11$

Out of all 9 pts tested

Abs. Max at

$(-1,2)$ and $(2,2)$

where

$f(-1,2) = f(2,2) = 18$.

Abs. Min at

$(-2,-2)$

where

$f(-2,-2) = -18$.

(b) Test interior of boundaries:

(i) $y = x$ $\quad f(x,x) = 9x$, $f'(x,x) = 9$ no crit. pts.

(ii) $x = 2$ $\quad f(2,y) = 2 - y^3 + 12y$, $f'(2,y) = 12 - 3y^2$

$f'(2,y) = 0$ when $y^2 = 4$. $f(2,2)$ already calculated

(iii) $y = 3$ $\quad \cancel{f(x,3) = x^3 - 3x + 10 \quad f'(x,3) = 3x^2 - 3}$

$\cancel{f'(x,3) = 0 \text{ when } x^2 = 1}$

$f(x,3) = x^3 - 3x + 9$, $f'(x,3) = 3x^2 - 3$

$f'(x,3) = 0$ when $x^2 = 1$, so crit. pts at $x = \pm 1$

$f(1,3) = 7$, $f(-1,3) = 11$

(iv) $x = -2$ $\quad f(-2,y) = -2 - y^3 + 12y$, $f'(-2,y) = 12 - 3y^2$

$f'(-2,y) = 0$ when $y^2 = 4$, so crit. pts at $y = \pm 2$

$f(-2,2) = 14$.

7. Calculate the double integral (10 points)

$$\iint_R \cos(x+2y)\,dA, \quad R = \{(x,y) \mid 0 \le x \le \pi, 0 \le y \le \pi/2\}.$$

$$\iint_R \cos(x+2y)\,dA = \int_0^{\pi}\int_0^{\pi/2} \cos(x+2y)\,dy\,dx$$

$$= \int_0^{\pi} \frac{1}{2}\sin(x+2y)\Big|_0^{\pi/2}\,dx$$

$$= \int_0^{\pi} \frac{1}{2}\left(\sin(x+\pi) - \sin(x)\right)dx$$

$$= \frac{1}{2}\Big[\cos x - \cos(x+\pi)\Big]_0^{\pi}$$

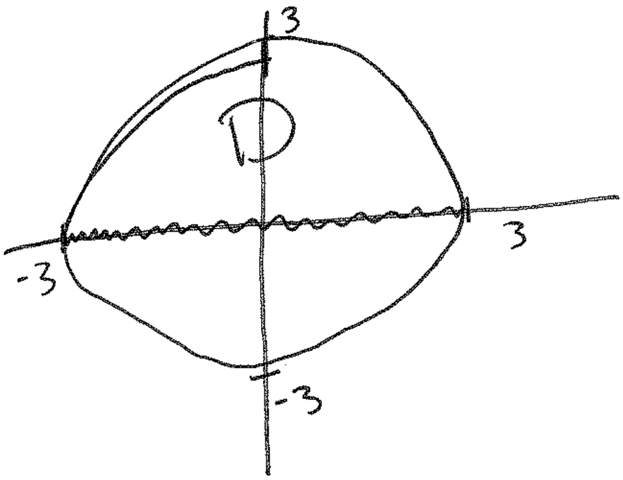
$$= \frac{1}{2}\Big[(-1-1) - (1 - -1)\Big] = -2$$

8. Consider the function given by $f(x, y) = xy + 2x - 3y$ and the region D bounded by the circle with center at the origin and radius 3 and the x-axis.

(a) Write the double integral $\iint_D f(x, y)dA$ as a Type I region. (5 points)

(b) Write the double integral $\iint_D f(x, y)dA$ as a Type II region. (5 points)

(c) Evaluate the double integral $\iint_D f(x, y)dA$ in whichever way you wish. (5 points)



(a) $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} xy+2x-3y\,dy\,dx$

(b) $\displaystyle\int_{0}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xy+2x-3y\,dx\,dy$

(c) $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} xy+2x-3y\,dy\,dx = \int_{-3}^{3}\left(\frac{xy^2}{2}+2xy-\frac{3y^2}{2}\right)\Big|_0^{\sqrt{9-x^2}}dx$

$\displaystyle = \int_{-3}^{3}\frac{x(9-x^2)}{2}+2x\sqrt{9-x^2}-\frac{3(9-x^2)}{2}\,dx$ Let $u = 9-x^2$, $du = -2x\,dx$

$\displaystyle = \int_{-3}^{3}\frac{1}{2}(9x-9x^3-27+3x^2)dx+\int -\sqrt{u}\,du$

$\displaystyle = \frac{1}{2}\left(\frac{9x^2}{2}-\frac{9x^4}{4}-27x+x^3\right)\Big|_{-3}^{3} - \frac{2}{3}u^{3/2}\Big|_{x=-3}^{x=3}$

$\displaystyle = \frac{1}{2}\left[\left(\frac{9\cdot3^2}{2}-\frac{9\cdot3^4}{4}-27\cdot3+3^3\right)-\left(\frac{9\cdot(-3)^2}{2}-\frac{9\cdot(-3)^4}{4}-27\cdot(-3)+(-3)^3\right)\right]-\frac{2}{3}\left[(9-3^2)^{3/2}-(9-(-3)^2)^{3/2}\right]$

$= 27-3\cdot27 = -54.$

Extra Credit

1. What is your favorite sports team? (15 points)

Game Cocks